UNIT IV

Linear equation with singular points

Introduction:

Consider the linear eqn with variable coefficient

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \dots + a_n(x)y = 0 \rightarrow (1)$$

We shall assume that the coefficients $a_0, a_1, \dots a_n$ are analytic at some point x_0.

⇒ A point x_0 such that $a_0(x_0) = 0$ is called a singular point of the (1)

⇒ A point x_0 is a regular singular point of (1) if the eqn can be written in the form,

$$(x-x_0)^n y^{(n)} + b_1(x)(x-x_0)^{n-1} y^{(n-1)} + \dots + b_n(x)y = 0 \rightarrow (2)$$

Near x_0, where the function $b_1 \dots b_n$ can be written in the form.

⇒ $b_k(x) = (x-x_0)^k B_k(x)$. $k = 1, 2, \dots n$

where $B_1, B_2 \dots B_n$ are analytic at x_0 Than (2) becomes

$$y^{(n)} + B_1(x)y^{(n-1)} + \dots + B_n(x)y = 0 \rightarrow (3)$$

Thus (2) is a generalization of the eqn with analytic coefficient

⇒ An eqn of the form,

$$c_0(x)(x-x_0)^n y^n + c_1(x)(x-x_0)^{n-1} y^{(n-1)} + \dots + c_n(x)y = 0$$

has a singular point at x_0, if $c_0, c_1 \dots c_n$ are analytic at x_0 and $c_0(x_0) \neq 0$.

This is because we may divide by $c_0(x)$ for x near x_0 to obtain an eqn of the form with

$$b_k(x) = \frac{c_k(x)}{c_0(x)}$$

It can be shown that b_k are analytic at x_0

The Euler eqn:

Theorem: 1

Consider the 2^{nd} order Euler equation $x^2y''+axy'+by=0$. (a,b) constants and the polynomial q given by

$$q(r)= r(r-1)+ar+b$$

[$q(r)=0$ is called the individual polynomial for 2^{nd} order Euler eqn]

A basis for the solns of the Euler equation on any interval not containing $x=0$ is given by $\varphi_1(x)=|x|^{r}$, $\varphi_2(x)=|x|^{r}\log|x|$, if r_1 is a root of q of multiplicity two.

Proof:

Let $L[y]= x^2y''+a_k xy'+by=0 \rightarrow (1)$

where a,b are constants.

Let us derive the soln on any interval not containing of point $x=0$.

Let $x>0$

Let $y=x^r \Rightarrow y'=rx^{r-1}$

$\frac{d}{dx}(a^x)=a^x\log a$

$L[x^r] = x^2[x(r-1)x^{r-2}]+a\,x[rx^{r-1}]+b.x^r=0$

$\Rightarrow r(r-1)x^r+arx^r+bx^r=0$

$\Rightarrow [r(r-1)+ar+b]x^r=0$

$L[x^r]=q(r).x^r=0 \rightarrow (2)$

where $q(r)=r(r-1)+ar+b=0$, but $x^r\neq 0$.

Let r_1,r_2 be roots of $q(r)$

If two roots are distinct.

(i) when $r_1\neq r_2$ Then the soln,

$\varphi_1=x^{r_1}$, $\varphi_2=x^{r_2}$

Suppose $r_1=r_2$ Then $q(r_1)=c_0$, $q'(r_1)=0$

Diff ① partially w.r.t 'r' weget,

$$\frac{\partial}{\partial r} L[x^r] = L\left[\frac{\partial}{\partial r} x^r\right]$$

$$= L[x^r \log x]$$

$$= [q'(r) + q(r) \log x] x^r$$

Now, if $r = r_1$, $L[x^r, \log x] = 0$

$\therefore$ $\varphi_2(x) = x^{r_1} \log x$ is a second soln associated with the roots r_1.

ii) for equal roots the soln are

$\varphi_1 = x^{r_1}$, $\varphi_2 = x^{r_1} \log x$, $x > 0$

In b.s the soln φ_1, φ_2 are L.I for $x > 0$

for consider constants c_1 and c_2 if $r_1 \neq r_2$ such that for $x > 0$

$$c_1 x^{r_1} + c_2 x^{r_2} = 0$$

Then $$c_1 + c_2 x^{r_2 - r_1} = 0 \rightarrow ③$$

Diff weget,

$$0 + c_2 (r_2 - r_1) x^{(r_2 - r_1) - 1} = 0$$

$\Rightarrow$ $c_2 = 0$ and ③ $\Rightarrow$ $c_1 = 0$ also. The soln are linearly independent.

If $r_1 = r_2$ and c_1, c_2 are constants $\ni$ for $x > 0$

$$c_1 x^{r_1} + c_2 x^{r_1} \log x = 0$$

Then $$c_1 + c_2 \log x = 0 \rightarrow ④$$

Diff w.r.t x weget.

$$0 + c_2 \frac{1}{x} = 0$$

$c_2 = 0$ since $x \neq 0$

④ $\Rightarrow$ $c_1 = 0$

$\therefore$ The soln are linearly independent.

Similarly let us consider the soln for $x<0$.

Consider $y = (-x)^r$ where r is a constant for $x<0$

$\Rightarrow$ $y' = -r(-x)^{r-1}$

and $y'' = r(r-1)(-x)^{r-2}$

Sub in $L[y] = x^2y'' + a_1(x)y' + by = 0$

we get,

$$x^2[r(r-1)(-x)^{r-2}] + ax[-r(-x)^{r-1}] + b(-x)^r = 0$$

$$\Rightarrow x^2 r(r-1)\frac{(-x)^r}{(-x)^2} + ax(-r)\frac{(-x)^r}{(-x)} + b(-x)^r = 0$$

$$\Rightarrow r(r-1)(-x)^r + ar(-x)^r + b(-x)^r = 0$$

$$\Rightarrow L[y] = q(r)((-x)^r), \quad x<0$$

where,

$$q(r) = r(r-1) + ar + b$$

ie) $L[(-x)^r] = q(r)(-x)^r, \quad x<0 \rightarrow (5)$

Also we have,

$$\frac{\partial}{\partial r}((-x)^r) = (-x)\log(-x), \quad x<0$$

where $q(r)=0$ has 2 roots of r_1 and r_2, $r_1 \neq r_2$ Then the soln are,

$$\phi_1(x) = (-x)^{r_1}, \quad \phi_2(x) = (-x)^{r_1}\log(-x), \quad x<0$$

Also,

ϕ_1 and ϕ_2 are linearly for $x<0$

Hence combining the soln, we get that on any interval containing the point $x=0$ the basis of soln are,

$\phi_1(x) = |x|^{r_1}$, $\phi_2(x) = |x|^{r_2}$, if $r_1 \neq r_2$

ie) n the distinct.

and $\phi_1(x) = |x|^{r_1}$ and $\phi_2(x) = |x|^{r_2}$ for x if $r_1 = r_2$

Generalized Euler eqn of n^{th} order

Extension of the result of Thm (1) of the Euler eqn of the n^{th} order]

Theorem: 2

Consider the Euler eqn $L[y] = x^n y^{(n)} + a_1 x^{n-1} y^{(n-1)} + \dots + a_n y = 0$,

where $a_1, a_2 \dots a_n$ are constants.

Now, $q(r) = r(r-1)(r-2) \dots (r-n+1) + a\, r(r-1)(r-2) \dots (r-n+2) + a_n$

is the identical polynomial for ①

Let $r_1, r_2 \dots r_s$ be the distinct roots of the identical polynomial q for ① and suppose $r_1, r_2 \dots r_n$ has multiply $m_1, m_2 \dots m_i$ then the n function.

$$|x|^{r_1}\quad |x|^{r_1}\log x \quad \dots \quad |x|^{r_1} \log^{m_1-1}|x|$$
$$|x|^{r_2}\quad |x|^{r_2}\log x \quad \dots \quad |x|^{r_2} \log^{m_2-1}|x|$$
$$\vdots$$
$$|x|^{r_s}\quad |x|^{r_s}\log x \quad \dots \quad |x|^{r_s} \log^{m_s-1}|x|$$

from a basis for the soln of them n^{th} order euler eqn ① on any interval not containing $x=0$

Proof:

Consider the eqn $L[y] = x^n y^n + a_1 x^{(n-1)} y^{(n-1)} + \dots + a_n y = 0 \rightarrow ①$

Where $a_1, a_2 \dots a_n$ are constants.

Let $y = x^r$ be a soln of ① for any constant r

$\Rightarrow \quad y' = r x^{r-1} \Rightarrow y'' = r(r-1)x^{r-2}$

$\vdots$

$y^n = r(r-1) \dots (r-n+1)x^{r-n}$

$\therefore \; L[y] = x^n [r(r-1) \dots (r-n+1)x^{r-n}] + a_1 x^{n-1}[r(r-1) \dots (r-n+2)x^{r-n+1}]$

$+ \dots + a_n] x^r = 0$

$\therefore \; L[|x|^r] = q(x)|x|^r$

Where $q(r)$ is (given by ② is statement)

$q(r) = [r(r-1) \dots (r-n+1)] + a_1[r(r-1) \dots (r-n+2)] + \dots + a_n$

This polynomial is called the indical eqn of the Euler eqn ①

Diff ③ w.r.t r, k times we get,

$$\frac{\partial^k}{\partial r^k} L[|x|^r] = L\left[\frac{\partial^k}{\partial r^k}|x|^r\right]$$

$$= L[|x|^r \log^k |x|]$$

$$= \frac{\partial^k}{\partial r^k}[q(x)x^r]$$

$\Rightarrow q^{(k)}(r).x^r + kq^{(k-1)} x^r \log x + \ldots + q(r)x^r(\log x)^k$

$\Rightarrow L[x^r \log x^k] = 0$

If r_1 is a root of multiplicity m_1 then $q(r_1)=0$, $q'(r_1)=0 \ldots q^{(m_1-1)}(r_1)=0$

$\Rightarrow |x|^{r_1}, |x|^{r_1}\log|x| \ldots |x|^{r_1}\log^{m_1-1}|x|$ are the soln of $L(y)=0$.

Repeating this process for each root of q we get the remaining soln.

Problem:

1.a) Find all soln of the following eqn for $x>0$.

$x^2y'' + 2xy' - 6y = 0$

Soln:

Given eqn is $x^2y'' + 2xy' - 6y = 0$

The indical polynomial q is given by

$q(r) = r(r-1) + 2r - 6 = (r+3)(r-2)$

$q(r) = 0 \Rightarrow r = 2, -3$ (both are distinct)

$\therefore$ The Euler soln is

$\varphi(x) = c_1|x|^{r_1} + c_2|x|^{r_2}$ if $r_1 \neq r_2$

$\therefore \varphi(x) = c_1x^{-3} + c_2x^2$, $x>0$, c_1, c_2 are constants

ie) $y = c_1x^{-3} + c_2x$

$y = \frac{c_1}{x^3} + c_2x$

1.b) $2x^2y''+xy'-y=0$

Soln:

Given eqn is $x^2y''+\frac{x}{2}y'-\frac{y}{2}=0 \rightarrow ①$

$q(r)= r(r-1)+ar+b=0$

$① \Rightarrow a=\frac{1}{2},\ b=-\frac{1}{2}$

$q(r)= r^2-r+\frac{r}{2}-\frac{1}{2}=0$

$r^2-\frac{r}{2}-\frac{1}{2}=0$

$$r=\frac{\frac{1}{2}\pm\sqrt{\frac{1}{4}-4(-\frac{1}{2})}}{2}$$

$$=\frac{\frac{1}{2}\pm\sqrt{\frac{1}{4}+2}}{2}$$

$$=\frac{\frac{1}{2}\pm\sqrt{\frac{9}{4}}}{2}$$

$$=\frac{\frac{1}{2}\pm\frac{3}{2}}{2}$$

$$=\frac{1\pm3}{4} = \frac{1+3}{4},\ \frac{1-3}{4}$$

$r=1,\ \frac{-1}{2}$

$\therefore y=\varphi(x)=c_1x+c_2x^{-1/2}$

1.c) $x^3y'''+2x^2y''-xy'+y=0$

Soln:

Given eqn is $x^3y'''+2x^2y''-xy'+y=0$

$q(r)= r(r-1)(r-2)+ar(r-1)+br+c=0$

Here, $a=2,\ b=-1,\ c=1$

$q(r)=r(r-1)(r-2)+2r(r-1)-r+1$

$=r(r^2-2r-r+2)+2r^2-2r-r+1$

$=r^3-2r^2-r^2+2r+2r^2-2r-r+1$

$q(r)=r^3-r^2-r+1$

$\Rightarrow r^2(r-1)-r+1=0$

$r^2(r-1)-(r-1)=0$

$(r-1)(r^2-1)=0$

$r-1=0, \quad r^2-1=0$

$r=1, \quad r=\pm 1$

$\therefore r=1,1,-1$

$\therefore \phi(x)=(c_1+c_2\log x)x+c_3x^{-1}$

where c_1, c_2, c_3 are constants

$\Rightarrow y=\phi(x)=c_1x+c_2x\log x+c_3/x$

1.d) $x^2y''-5xy'+9y=x^3, \quad x>0$

Soln:

Given eqn is $x^2y''-5xy'+9y=x^3$, $x>0$.

Indical polynomial is

$q(r)=r(r-1)-5r+9=0$

$r^2-r-5r+9=0$

$r^2-6r+9=0$

$(r-3)(r-3)=0$

$r=3,3$

$\phi_1(x)=x^3, \quad \phi_2(x)=x^3\log x$

$\therefore y=\phi(x)=c_1x^3+c_2x^3\log x$

Since $x=3$ is a root of multiplicity two a particular soln ψ of $L[y]=x^3$ is given by

$$\psi(x)=\frac{1}{q''(3)}x^3(\log x)^2=\frac{x^3}{2}\log^2 x$$

$\therefore$ most general soln is

$\phi(x)=c_1x^3+c_2x^3\log x+\frac{x^3}{2}\log^2 x.$

1.e) $x^2y''-3xy'+5y=0$

Soln:

Given $x^2y''-3xy'+5y=0 \rightarrow (1)$

Now, $q(r)=r(r-1)+ar+b=0$

$a=-3, b=5$

$= \frac{4 \pm \sqrt{16-4(5)}}{2} = \frac{4 \pm \sqrt{16-20}}{2}$

$= \frac{4 \pm \sqrt{-4}}{2} = \frac{4 \pm 2i}{2}$

$= 2 \pm i$

$q(r) = r^2 - 4r + 5 = 0$

$r = 2 \pm i$

$\psi = \varphi(x) = C_1 |x|^{2+i} + C_2 |x|^{2-i}$

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1.f) $x^2 y'' + xy' + y = 0$, for $x \neq 0$.

Soln:

The polynomial q is given by

$q(r) = r(r-1) + r + 1$

$= r^2 - r + r + 1$

$q(r) = r^2 + 1$

$q(r) = 0$.

$r^2 + 1 = 0$

$r = \pm i$

$\varphi_1(x) = |x|^i$, $\varphi_2(x) = |x|^{-i}$, $x \neq 0$.

Where $|x|^i = e^{i \log |x|}$

Note:

Another basis φ_1, φ_2 is

$\varphi_1(x) = \cos[\log |x|]$

$\varphi_2(x) = \sin[\log |x|]$, $x \neq 0$.

Working rule to find the P.I of $L[y] = x^k$

Case (i)

k is not a root of indical eqn.

$q(r) = 0$, ie) $q(k) \neq 0$

Then $P.I = \psi(x) = \frac{x^k}{q(k)}$

case (ii)

k is a root of multiplicity one for $q(r) = 0$ Then

$P.I = \psi(x) = \frac{x^k \log x}{q'(x)}$

Case (iii)

k is a root of multiplicity two for $q(r)=0$ Then

$$P.I = \psi(x) = \frac{x^k (\log x)^2}{q''(k)}$$

Case (iv)

k is a root of multiplicity m for $q(r)=0$ Then

$$P.I = \psi(x) = \frac{x^k (\log x)^m}{q^m(k)}$$ (General case)

1.g) $x^2y'' - 3xy' + 4y = 0$

Soln:

$q(r) = r(r-1) + ar + b = 0$

$a = -3, \; b = 4$

$r^2 - r - 3r + 4 = 0$

$r^2 - 4r + 4 = 0$

$(r-2)(r-2) = 0$

$r = 2, 2$

$\phi(x) = c_1 |x|^2 + c_2 |x|^2 \log x$

2.a) Find all soln of the following eqn for $|x| > 0$

$x^2y'' + xy' + 4y = 0$

Soln:

Given eqn is $x^2y'' + xy' + 4y = 1$

Then indical Polynomial is

$q(r) = r(r-1) + ar + b = 0$

Here $a = 1, \; b = 4$

$\Rightarrow q(r) = r^2 - r + r + 4 = 0$

$r^2 + 4 = 0$

$r^2 = -4$

$r = \pm 2i$

$\phi_1(x) = |x|^{2i}, \quad \phi_2(x) = |x|^{-2i}$

$\therefore L[y] = c_1 |x|^{2i} + c_2 |x|^{-2i}$

To find particular soln ψ

$$L(y) = 1 = x^0$$

$$\psi = \frac{1}{q(k)} x^k = \frac{1}{q(0)} x^0 = \frac{1}{4}$$

since $x^k = 1 = x^0$ and $q(0) = 4$

$$\therefore \phi = c_1 |x|^{2i} + c_2 |x|^{-2i} + \frac{1}{4}$$

2.b) $x^2 y'' + xy' - 4\pi y = x$

Soln:

$$q(r) = r(r-1) + r - 4\pi = 0$$

$$r^2 - 4\pi = 0$$

$$r^2 = 4\pi$$

$$r = \pm 2\sqrt{\pi}$$

$$\therefore \phi(x) = c_1 |x|^{2\sqrt{\pi}} + c_2 |x|^{-2\sqrt{\pi}}$$

To find P.I, ψ

Let $\phi(x)$ be any soln of $L[y] = x$

$$q(k) = q(1) \neq 0$$

$$= \frac{x}{1 - 4\pi} \qquad [\because q(x) = r^2 - 4\pi \Rightarrow q(1) = 1 - 4\pi]$$

$\therefore$ General soln is

$$\phi(x) = c_1 |x|^{2\sqrt{\pi}} + c_2 |x|^{-2\sqrt{\pi}} + \frac{1}{1-4\pi} x$$

2. c) $x^2 y'' + xy' - 4y = x$

Soln:

$$q(r) = r(r-1) + r - 4$$

$$r^2 - r + r - 4 = 0$$

$$r^2 - 4 = 0$$

$$r = \pm 2$$

$$\therefore \phi(x) = c_1 |x|^2 + c_2 |x|^{-2}$$

To find P.I, ψ

$$L[y] = x = x^1$$

1 is not a root of indical eqn

$$q(r) = 0, \quad q(1) \neq 0, \quad q(1) = -3$$

$\therefore \psi(x) = \frac{x^k}{q(k)} = \frac{x^1}{-3} = \frac{x}{-3}$

$\therefore$ General soln is $y = c_1|x|^2 + c_2|x|^{-2} - \frac{|x|}{3}$

Problem:

Let $L[y] = x^2y'' + axy' + by$, where, a, b are constants and.

Let q be the polynomial $q(r) = r(r-1) + ar + b$

(a) S.T the eqn $L[y] = x^k$ has a soln ψ of the form,

$\psi(x) = cx^k$ if $q(k) \neq 0$, Compute c.

(b) Suppose k is a root of q of multiplicity one S.T there is a soln ψ of $L[y] = x^k$ if the form, $\psi(x) = cx^k \log x$, compute c.

(c) find a soln of $L[y] = x^k$ in case k is a double root of q.

Soln:

$L[y] = x^2y'' + axy' + by$

and $q(r) = r(r-1) + ar + b$

(a) Let $\psi(x) = cx^k$ be a soln of $L[y] = x^k$

$\therefore L[x^k] = x^k$

$L[cx^k] = [x^k]$

$= c[x^2(x^k)'' + ax(x^k)' + bx^k]$

$= c[x^2 k(k-1)x^{k-2} + a.xkx^{k-1} + bx^k]$

$= c[k(k-1) + ak + b]$

$L[cx^k] = c\,q(k)x^k$

Since $L[cx^k] = x^k$

We have,

$cq(k)x^k = x^k$ since $x \neq 0$.

$\Rightarrow cq(k) = 1$

$\Rightarrow c = \frac{1}{q(k)}$

$\psi(x) = c x^k$

Then $L[c x^k \log x] = c[q'(x) + q(x)\log x] x^k = x^k$ (13)

$L[\psi] = x^k$

$\therefore \; c\,q'(k) = 1$ since $q(k) = 0$

$\Rightarrow c = \dfrac{1}{q'(k)}$

$\therefore \psi(x) = \dfrac{1}{q'(k)} x^k \log x$

c) Now k is a root of multiplicity two

Let $\psi(x) = c x^k (\log x)^2$ since $L[\psi] = x^k$

$\therefore L[c x^k (\log x)^2] = x^k$

$\Rightarrow c[q''(k) + 2q'(k)\log x + q(x)\log^2 x]\log^2 x = 1$

since $q(k) = 0$, $q'(k) = 0$

$\therefore c\,q'(k) = 1 \Rightarrow c = \dfrac{1}{q'(k)}$

But $q(k) = r^2 - r + ar + b$

$q''(k) = 2$

$\psi(x) = \dfrac{1}{q''(k)} x^k \log^2 x$

$\psi(x) = \dfrac{x^k}{2} \log^2 x$

Method of finding the nature of singular points.

Method:1

Step:1

Let $a_0(x) y^{(n)} + a_1(x) y^{(n-1)} + \dots + a_n(x) y = 0$

The given eqn

Put $a_0(x) = 0$ and find its roots. There are the singular points of the given eqn.

Step: 2

Let $x = x_0$ be a singular point, (14)

Rewrite the given eqn in the form,

$$(x-x_0)^n y^n + b_1(x)(x-x_0)^{(n-1)} y^{(n-1)} + \dots + b_n(x) y_n = 0$$

If all $b_1(x) + b_2(x) + \dots + b_n(x)$ are analytic at $x = x_0$ Then $x = x_0$ is a regular singular point. Otherwise it is an irregular singular point.

Step: 3

Repeat Step (3) to test the nature of the other singular Poin given by $a_0(x) = 0$

Method - II

Step: 1

As in case method I.

Determine the singular points by setting $a_0(x) = 0$.

If the difference eqn is given of the form,

$$y^{(n)} + P_1(x) y^{(n-1)} + P_2(x) y^{(n-2)} + \dots + P_n(x) y = 0$$

The singular points are given by

$$P_1(x) = \alpha \text{ (or) } \frac{1}{P_1(x)} = 0$$

Step: 2

Rewrite in the given eqn in the form

$y^{n}(x) + P_1(x) y^{(n-1)}(x) + P_2(x) y^{(n-2)}(x) + \dots + P_n(x)$ are with analytic at $x = x_0$.

If this is so Then $x = x_0$ is a regular singular Points. Otherwise it is an irregular singular point.

xistence of Power Series Soln.

dinary Point:

If all the coefficient $a_0(x), a_1(x) \ldots a_n(x)$ of the given D.E are analytic at $x=x_0$ then this point is called an ordinary point

we consider only IInd order eqn of the form.

$$a_0(x)\frac{d^2y}{dx^2}+a_1(x)\frac{dy}{dx}+a_2(x)y=0$$

(or)

$$\frac{d^2y}{dx^2}+P(x)\frac{dy}{dx}+Q(x)y=0$$

At an ordinary point this eqn has two linearly power series soln.

Regular Singular Point:

At a regular singular point the given IInd order D.E has atleast powers series soln, the existence of the second power series soln depends on the nature of the roots of the indical eqn.

Irregular singular point:

The given D.E does not have any power series soln at an irregular singular point.

Problem:

3.a) Find the singular points of the following eqn and determine those which are regular singular points $x^2y''+(x+x^2)y'-y=0$

Soln:

Given eqn is $x^2y''+(x+x^2)y'-y=0 \rightarrow (1)$

writing the eqn of the form.

$$\frac{d^2y}{dx^2}+P(x)\frac{dy}{dx}+Q(x)=0$$

We get,

$$\frac{d^2y}{dx^2} + \frac{1+x}{x}\frac{dy}{dx} - \frac{1}{x^2}y = 0$$

We get,

$$P(x) = \frac{1+x}{x}, \quad Q(x) = \frac{-1}{x^2}$$

At $x=0$, $P(x) = \infty \Rightarrow x=0$ is a Singular

Now.

$$xP(x) = 1+x, \quad \text{and} \quad x^2Q(x) = -1$$

These are analytic at $x=0$

$\Rightarrow$ $x=0$ is a regular singular points

3.b) $(1-x^2)y'' - 2xy' + 2y = 0$

Soln:

write the eqn in the form.

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0$$

we get.

$$\frac{d^2y}{dx^2} - \frac{2x}{1-x^2}\frac{dy}{dx} + \frac{2}{1-x^2}y = 0$$

$$\Rightarrow P(x) = -\frac{2x}{1-x^2}, \quad Q(x) = \frac{2}{1-x^2}$$

Nature of singular point at $x=1$

$$(x-x_0)P(x) = (x-1)P(x) = \frac{2x}{1+x}$$

$$(x-x_0)^2Q(x) = (x-1)^2Q(x) = \frac{2(1-x)}{1+x}$$

These are analytic diff at $x=1$

$\Rightarrow$ $x=1$ is a regular singular point.

$x=-1$

$$(x-x_0)P(x) = (x+1)P(x) = \frac{2x}{x-1}$$

$$(x-x_0)^2Q(x) = (x+1)^2Q(x) = \frac{2(1+x)}{1-x}$$

These are analytic at $x=-1$

$\Rightarrow$ $x=-1$ is a regular Singular Point.

) $(x^2+x-2)^2 y'' + 3(x+2)y' + (x-1)y = 0$

Soln:

Given eqn is $(x^2+x-2)^2 y'' + 3(x+2)y' + (x-1)y = 0$

singular Points are given by.

$(x^2+x-2)^2 = 0 \Rightarrow x^2+x-2 = 0$

$(x+2)(x-1) = 0$

$\Rightarrow x = -2, 1$ are the singular Points

Nature of singular Points at $x=1$

The given D.E can be written as

$(x-1)^2(x+2)^2 y'' + 3(x+2)y' + (x-1)y = 0$

$\Rightarrow (x-1)^2 y'' + \frac{3}{x+2} y' + \frac{x-1}{(x+2)^2} y = 0$

$\Rightarrow (x-1)^2 y'' + \frac{3}{(x+2)(x-1)}(x-1)y' + \frac{(x-1)}{(x+2)^2} y = 0$

Comparing with

$(x-x_0)^2 y'' + (x-x_0)a_1(x)y' + a_2(x)y = 0$

taking $x_0 = 1$ weget,

$a_1(x) = \frac{3}{(x+2)(x-1)}$ and $a_2(x) = \frac{x-1}{(x+2)^2}$

$a_1(x)$ is not analytic singular point

$\therefore$ the given D.E is

$x = -2$

$(x-1)^2(x+2)^2 y'' + 3(x+2)y' + (x-1)y = 0$

$\Rightarrow (x+2)^2 y'' + \frac{3(x+2)}{(x-1)^2} y' + \frac{1}{x-1} y = 0$

comparing with $(x-x_0)^2 y'' + (x-x_0)a_1(x)y' + a_2(x)y = 0$ taking $x_0 = -2$

weget,

$a_1(x) = \frac{3}{(x-1)^2}$, $a_2(x) = \frac{1}{x-1}$

There are analytic at $x=2$

$\Rightarrow x = -2$ is a regular singular point.

3.d) $3x^3 y'' + x^6 y' + 2xy = 0$

3.e) $x^2 y'' - 5y' + 3x^2 y = 0$

3.f) $xy'' + 4y = 0$

3.g) $x^2 y'' + (\sin x)y' + (\cos x)y = 0$

Step: 1

Assume that soln in the form

$$\phi(x) = x^r \sum_{k=0}^{\infty} c_k x^k$$

Find all the derivatives $\phi'(x), \phi''(x)$ and substitute the given eqn.

Collect the coefficient of lower most power of x namely x^r.

That is called the ~~individual~~ indicial eqn of $q(r)=0$.

Step: 2

Find the roots r_1 and r_2 of the ~~individual~~ indicial eqn $q(r)=0$.

Case (i)

If the roots r_1 and r_2 are distinct and do not differ by an integer (ie) $r_1 \neq r_2$, $r_1 - r_2 \neq$ an integer. Then there two I.P power series soln is given by

$$\phi_1(x) = x^{r_1} \sum_{k=1}^{\infty} c_k x^k \quad \text{and} \quad \phi_2(x) = x^{r_2} \sum_{k=1}^{\infty} c_k x^k$$

Case (ii)

If the roots r_1 and r_2 are distinct and differ by an integer,

ie) $r_1 \neq r_2$ and $r_1 - r_2$ an integer,

(a) Let $r_1 > r_2$

If none of the coefficients of $\phi(x)$ become α on the substitution of $r = r_1$ then

$$\phi_1(x) = x^{r_2} \sum_{k=1}^{\infty} c_k x^k$$

This will contain two arbitrary constants the second soln $\phi_2(x)$ obtained by putting $r = r_1$ merly contains a numerical multiple one of the power series contained in the first soln and Hence rejected.

) let $r_1 > r_2$

If some of the coefficients of $\phi(x)$ become ∞ on substituting $r = r_2$ in $\phi(x)$. Then we put $c_0 = d_0(r - r_2)$ and then put $r = r_2$ to obtain the first soln of $\phi_1(x)$ to get the second soln $\phi_2(x)$. we put $r = r_2$ in $\frac{\partial \phi}{\partial r}$

ie) $\phi_2(x) = \left(\frac{\partial \phi}{\partial r}\right)_{r=r_2}$

Case (iii)

Roots of the indi~~vidual~~cial eqn are equal ie) $r_1 = r_2$

In this case the two soln are given by

$\phi_1(x) = x^{r_1} \sum_{k=1}^{\infty} c_k x^k$ and $\phi_2(x) = \left(\frac{\partial \phi}{\partial r}\right)_{r=r_1}$

The second soln always consists of the product of the I^{st} soln (or) numerical multiple of it and $\log x$ added to another series

Result:

The indi~~vidual~~cial eqn of $x^2 y'' + a(x) x y' + b(x) y = 0 \rightarrow (1)$ can be easily obtained by I^{st} comparing to eqn given in the problem with (1) there by identifying $a(x)$ and $b(x)$ then indicial eqn is given by

$q(r) = r(r-1) + a(0) - r + b(0)$

Problem:

4.a) Compute the indicial polynomial and there roots of the following eqn $x^2 y'' + (x + x^2) y' - y = 0 \rightarrow (1)$

Soln:

$L[y] = x^2 y'' + (x + x^2) y' - y = 0$

$x = 0$ is a regular singular point

let $\phi(x) = \sum_{k=0}^{\infty} c_k x^{r+k}$

$\phi'(x) = \sum_{k=0}^{\infty} c_k (r+k) x^{r+k-1}$

$\phi''(x) = \sum_{k=0}^{\infty} c_k (r+k)(r+k-1) x^{r+k-2}$

$$\therefore (1) \Rightarrow \left. \begin{aligned} & x^2 \sum_{k=0}^{\infty} c_k (r+k)(r+k-1) x^{r+k-2} \\ & + (x+x^2) \sum_{k=0}^{\infty} c_k (r+k) x^{r+k-1} - \sum_{k=0}^{\infty} c_k x^{r+k} \end{aligned} \right\} = 0$$

$$\Rightarrow \left. \begin{aligned} & \sum_{k=0}^{\infty} c_k (r+k)(r+k-1) x^{r+k} + \sum_{k=0}^{\infty} c_k (r+k) x^{r+k} \\ & + \sum_{k=0}^{\infty} c_k (r+k) x^{r+k+1} - \sum_{k=0}^{\infty} c_k x^{r+k} \end{aligned} \right\} = 0$$

$$\Rightarrow \left. \begin{aligned} & \sum_{k=0}^{\infty} c_k (r+k)(r+k-1) x^{r+k} + \sum_{k=0}^{\infty} c_k (r+k) x^{r+k} \\ & + \sum_{k=1}^{\infty} c_{k-1} (r+k-1) x^{r+k} - \sum_{k=0}^{\infty} c_k x^{r+k} \end{aligned} \right\} = 0$$

$$\Rightarrow \left. \begin{aligned} & c_0 (r)(r-1) x^r + \sum_{k=1}^{\infty} c_k (r+k)(r+k-1) x^{r+k} \\ & + c_0 r x^r + \sum_{k=1}^{\infty} c_k (r+k) x^{r+k} + \sum_{k=1}^{\infty} c_{k-1} (r+k-1) x^{r+k} \\ & - c_0 x^r + \sum_{k=0}^{\infty} c_k x^{r+k} \end{aligned} \right\} = 0$$

coefficients of powers,

$$x^r \Rightarrow [c_0 r(r-1) + c_0 r - c_0] = 0$$

$$\Rightarrow c_0 [r^2 - r + r - 1] = 0$$

$$\Rightarrow r^2 - 1 = 0 \qquad \text{since } c_0 \neq 0.$$

$$\Rightarrow r = \pm 1$$

$\therefore$ the roots are $r_1 = 1$, $r_2 = -1$

Aliter:

$$\varphi(x) = c_0 x^r + c_1 x^{r+1} + c_2 x^{r+2} + \dots$$

$$\varphi'(x) = c_0 r x^{r-1} + c_1 (r+1) x^r + c_2 (r+2) x^{r+1} + \dots$$

$$\varphi''(x) = c_0 r(r-1) x^{r-2} + c_1 r(r+1) x^{r-1} + c_2 (r+1)(r+2) x^{r+2} + \dots$$

Hence,

$$x^2 \varphi''(x) = c_0 r(r-1) x^r + c_1 (r+1) r x^{r+1} + c_2 (r+2)(r+1) x^{r+2} + \dots$$

$$(x+x^2) \varphi'(x) = c_0 r x^r + c_1 (r+1) x^{r+1} + \dots + c_0 r x^{r+1} + c_1 (r+1) x^{r+1} + \dots$$

$$-\varphi(x) = -c_0x^r - c_1x^{r+1} - c_2x^{r+2} \dots$$

Adding we get,

$$L[\varphi(y)] = c_0[r(r-1)+r-1]x^r + [(r+1)r+(r+1)-1]c_1 + c_0x^{r+1} + \dots$$

equating the coefficient of the lower powers of x (namely coefficient of x^r) to zero, we get the ~~individual~~ indicial polynomial for the given eqn as

$r^2-1=0$ & the roots are $r_1=1,\ r_2=-1$

Second order eqn with regular singular points

A second order eqn with regular singular points at x_0 has the form,

$$(x-x_0)^2y'' + a(x)(x-x_0)y' + b(x)y = 0 \quad \rightarrow (1)$$

where a, b are analytic at x_0.

Thus a, b have power series expansion

$$a(x) = \sum_{k=0}^{\infty} \alpha_k (x-x_0)^k$$

$$b(x) = \sum_{k=0}^{\infty} \beta_k (x-x_0)^k$$

which are convergent on some interval $|x-x_0| < r_0$ for some $r_0 > 0$

If $x_0 \neq 0$ we can change (1) to an equivalence eqn with a regular singular point at the origin.

Let $t = x - x_0$ and

$$\bar{a}(t) = a(x_0+t) = \sum_{k=0}^{\infty} \alpha_k t^k$$

$$\bar{b}(t) = b(x_0+t) = \sum_{k=0}^{\infty} \beta_k t^k$$

The power series for $\bar{a}, \bar{b}$ converges on the interval $|t| < r_0$ about $t=0$

Let φ be the any so

Define $\bar{\varphi}(t) = \bar{\varphi}(x_0+t)$

Then $\frac{d\bar{\varphi}(t)}{dt} - \frac{d\varphi}{dx}(x_0+t)$

$$\frac{d^2\bar{\varphi}(t)}{dt^2} = \frac{d^2\varphi}{dx^2}(x_0+t)$$

we see that $\bar{\varphi}$ satisfies,

$$t^2u'' + \bar{a}(t)u' + \bar{b}(t)u = 0 \rightarrow ②$$

where $u' = \frac{du}{dt}$ This is an eqn with a regular singular point at $t=0$.

Conversely.

If $\bar{\varphi}$ satisfies ② The fun φ given by

$\varphi(x) = \bar{\varphi}(x-x_0)$ satisfies ①

In this case ② is equivalent to ① with $x_0 = 0$ in ①.

we write ① as

$$L(y) = x^2y'' + a(x)y' + b(x)y = 0 \rightarrow ③$$

where a, b are analytic at the origin and have power series expansions.

$$a(x) = \sum_{k=0}^{\infty} \alpha_k x^k$$

$$b(x) = \sum_{k=0}^{\infty} \beta_k x^k$$

which are convergent on $|x| < r_0$, $r_0 > 0$

The Euler eqn is a special case of ③. where a, b constant.

Note:

The coefficient of the higher order terms (terms with x as a factor) in the series ④ is to introduce series into the soln of ③

b) $x^2y'' + xy' + (x^2 - \frac{1}{4})y = 0$

Soln:

$$L(y) = x^2y'' + xy' + (x^2 - \tfrac{1}{4})y = 0 \quad \rightarrow (1)$$

$\phi(x)$, $\phi'(x)$ and $\phi''(x)$ as above (1)

$$\left.\begin{aligned}&\sum_{k=0}^{\infty} C_k(r+k-1)(r+k)x^{r+k} + \sum_{k=0}^{\infty} C_k(r+k)x^{r+k}\\ &+ \sum_{k=0}^{\infty} C_k x^{r+k+2} - \frac{1}{4}\sum_{k=0}^{\infty} C_k x^{r+k}\end{aligned}\right\} = 0$$

$$\Rightarrow \left.\begin{aligned}&\sum_{k=0}^{\infty} C_k(r+k-1)(r+k)x^{r+k} + \sum_{k=0}^{\infty} C_k(r+k)x^{r+k}\\ &+ \sum_{k=2}^{\infty} C_{k-2}x^{r+k} - \frac{1}{4}\sum_{k=0}^{\infty} C_k x^{r+k}\end{aligned}\right\} = 0$$

$$\left.\begin{aligned}&C_0(r-1)rx^r + C_1r(r+1)x^{r+1} + \sum_{k=2}^{\infty} C_k(r+k-1)(r+k)x^{r+k}\\ &+ C_0rx^r + C_1(r+1)x^{r+1} + \sum_{k=2}^{\infty} C_k(r+k)x^{r+k} + \sum_{k=2}^{\infty} C_{k-2}x^{r+k}\\ &- \frac{1}{4}C_0x^r - \frac{1}{4}C_1x^{r+1} - \frac{1}{4}\sum_{k=2}^{\infty} C_k x^{r+k}\end{aligned}\right\} = 0$$

equating coefficient of x^r

$$C_0(r-1)r + C_0r - \frac{1}{4}C_0 = 0$$

$$r^2 - r + r - \frac{1}{4} = 0$$

$$r^2 - \frac{1}{4} = 0$$

$$4r^2 - 1 = 0$$

$$r^2 = \frac{1}{4}$$

$$r = \pm\frac{1}{2}$$

The roots are $r_1 = \frac{1}{2}$ and $r_2 = -\frac{1}{2}$

4.c) $4x^2y'' + (4x^4 - 5x)y' + (x^2+2)y = 0$

4.d) $x^2y'' + (x - 3x^2)y' + e^xy = 0$

4.e) $x^2y'' + (\sin x)y' + (\cos x)y = 0$

4.f) $x^2y'' + (x - x^2)y' + y = 0$

1. Find a soln φ of the form $\varphi(x) = x^r \sum_{k=0}^{\infty} c_0 x^k$, $(x>0)$ for the eqn $x^2y'' + \frac{3}{2}xy' + xy = 0$

Soln:

$$L(y) = x^2y'' + \frac{3}{2}xy' + xy = 0 \rightarrow ①$$

① has a regular singular point at the origin.

Let $\varphi(x) = x^r \sum_{k=0}^{\infty} c_k x^{r+k} = c_0x^r + c_1x^{r+1} + c_2x^{r+2} + \dots$

$$\varphi'(x) = c_0 r x^{r-1} + c_1(r+1)x^r + c_2(r+2)x^{r+1} + \dots$$

$$\varphi''(x) = c_0 r(r-1)x^{r-2} + c_1(r+1)r x^{r-1} + c_2(r+2)(r+1)x^r + \dots$$

and hence,

$$x^2\varphi''(x) = c_0 r(r-1)x^r + c_1(r+1)r x^{r+1} + c_2(r+2)(r+1)x^{r+2} + \dots$$

$$\frac{3}{2}x\varphi'(x) = \frac{3}{2}c_0 r x^r + \frac{3}{2}c_1(r+1)x^{r+1} + \frac{3}{2}c_2(r+2)x^{r+2} + \dots$$

$$x\varphi(x) = c_0x^{r+1} + c_1x^{r+2} + \dots$$

Adding we obtain,

$$L[\varphi(x)] = [r(r-1) + \tfrac{3}{2}r]c_0x^r + \{[(r+1)r + \tfrac{3}{2}(r+1)]c_1 + c_0\}x^{r+1}$$
$$+ \{[(r+2)(r+1) + \tfrac{3}{2}(r+2)]c_2 + c_1\}x^{r+2} + \dots \longrightarrow ②$$

The indicial polynomial is obtained by equating to zero the coefficients of lowest power of x

$$[r(r-1) + \tfrac{3}{2}r] + c_0 = 0$$

$$r(r-1) + \tfrac{3}{2}r = 0 \qquad \because c_0 \neq 0$$

$$\Rightarrow r^2 - r + \tfrac{3}{2} + r = 0$$

$$r^2 + \tfrac{1}{2}r = 0$$

$$r(r + \tfrac{1}{2}) = 0$$

$$r = 0, \quad r = -\tfrac{1}{2}$$

The two roots are $r_1 = 0,\ r = -\frac{1}{2}$

② $\Rightarrow$ $L[\varphi(x)] = q(r)c_0x^r + x^r \sum_{k=1}^{\infty} [q(r+k)c_k + c_{k-1}]x^k$

where $q(r) = r(r-1) + \frac{3}{2}r$

$= r(r+\frac{1}{2})$

since $L[\varphi(x)] = q(r)c_0x^r + [q(r+1)c_1 + c_0]x^{r+1} + [q(r+2)c_2 + c_1]x^{r+2} + \dots$

$= q(r)c_0x^r + x^r \sum_{k=1}^{\infty} [q(r+k)c_k + c_{k-1}]x^k$

since $L[\varphi(x)] = 0$

The above power series variables

$\therefore q(r+k)c_k + c_{k-1} = 0$

$\therefore c_k = \dfrac{-c_{k-1}}{q(r+k)}, \quad k = 1, 2 \dots$

$= \dfrac{-c_{k-1}}{(r+k)(r+k+\frac{1}{2})}$

Thus we get,

$$c_k = \frac{-(-1)^k c_0}{q(r+k)\,q(r+k-1)\dots q(r+1)}$$

if $r_1 = 0$, $q(r_1+k) = q(k) \neq 0$ for $k = 1, 2 \dots$

if $r_2 = \frac{-1}{2}$, $q(r_2+k) = q(\frac{-1}{2}+k) \neq 0$, for $k = 1, 2 \dots$

Let $c_0 = 1$, $r = r_1 = 0$, we obtain a soln φ given by

$$\varphi_1(x) = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k x^k}{q(k)\,q(k-1)\dots q(1)}$$

Taking $c_0 = -1$, $r = r_2 = \frac{-1}{2}$ we get,

another soln φ_2 given by

$$\varphi_2(x) = x^{-1/2} + x^{-1/2} \sum_{k=1}^{\infty} \frac{(-1)^k x^k}{q(k-\frac{1}{2})\,q(k-3/2) - q(1/2)}$$

Thus φ_1, φ_2 are soln of ① $\forall\, x > 0$.

Find a soln $\varphi(x) = x^r \sum_{k=0}^{\infty} c_k x^k$ $(x > 0)$ for the eqn

$2x^2y'' + (x^2 - x)y' + y = 0$

Soln:

Let $L[y] = 2x^2y'' + (x^2 - x)y' + y = 0 \rightarrow$ ①

Let $\varphi(x) = x^r \sum_{k=0} c_k$ ...

$$\varphi(x) = \sum_{k=0}^{\infty} c_k x^{k+r} \rightarrow ②$$

$$\varphi'(x) = \sum_{k=0}^{\infty} c_k (k+r) x^{k+r-1}$$

$$\varphi''(x) = \sum_{k=0}^{\infty} c_k (k+r)(k+r-1) x^{k+r-2}$$

$$L(\varphi(x)) = 2x^2 \left[\sum_{k=0}^{\infty} c_k (k+r)(k+r-1) x^{k+r-2} \right] + (x^2 - x) \sum_{k=0}^{\infty} c_k (k+r) x^{k+r-1}$$
$$+ \sum_{k=0}^{\infty} c_k x^k$$

$$= 0$$

$$\Rightarrow \left. \begin{array}{l} \sum_{k=0}^{\infty} c_k \, 2(k+r)(k+r-1) + \sum_{k=0}^{\infty} c_k (k+r) x^{k+r+1} \\ - \sum_{k=0}^{\infty} c_k (k+r) x^{k+r} + \sum_{k=0}^{\infty} c_k (k+r) \end{array} \right\} = 0$$

$$\text{i.e.) } \left. \begin{array}{l} \sum_{k=0}^{\infty} [2(k+r)(k+r-1) - (k+r+1)] c_k x^{k+r} \\ + \sum_{k=0}^{\infty} c_k (k+r) x^{k+r+1} \end{array} \right\} = 0 \rightarrow ③$$

equating the coefficients of lowest power of x namely x^r to weget,

$$2r(r-1) - r + 1 = 0$$

$$\Rightarrow 2r^2 - 3r + 1 = 0$$

Weget, $r_1 = 1, \quad r_2 = 1/2$

Now, ③

$$\Rightarrow \sum_{k=0}^{\infty} \{[2(k+r)(k+r-1) - (k+r)+1] c_k + c_{k-1}(k+r-1)\} x^{r+k} = 0$$

This is possible when each coefficient of $x = 0$

$$[2(k+r)(k+r-1) - (k+r)+1] c_k + c_{k-1}[k+r-1] = 0$$

$$\Rightarrow [2(k+r)^2 - 3(k+r) + 1] c_k + (k+r-1) c_{k-1} = 0$$

$$(k+r-1)(2k+2r-1) c_k = (-(k+r-1)) c_{k-1}$$

$$c_k = \frac{-1}{2k+2r-1} c_{k-1}, \quad k = 1, 2 \ldots$$

for $r = r_1 = 1$, $c_k = \frac{-1}{2k+1} c_{k-1}$, $k = 1, 2 \ldots$

$k = 1$, $\Rightarrow c_1 = -\frac{1}{3} c_0$

$k = 2 \Rightarrow c_2 = -\frac{1}{5} c_1 = \frac{(-1)^2}{3 \cdot 5} c_0$

$k = 3 \Rightarrow c_3 = -\frac{1}{7} c_2 = \frac{(-1)^3}{3 \cdot 5 \cdot 7} c_0$ etc.

$\therefore \varphi_1(x) = x \sum_{k=0}^{\infty} \frac{(-1)^k x^k c_0}{1 \cdot 3 \cdot 5 \cdot 7 \ldots (2k+1)}$

for $r = r_2 = \frac{1}{2}$, $c_k = \frac{-1}{2k} c_{k-1}$, $k = 1, 2 \ldots$

$k = 1$, $c_1 = -\frac{1}{2} c_0$

$k = 2$, $c_2 = -\frac{1}{4} c_1 = (-1)^2 \frac{1}{2 \cdot 4} c_0 = \frac{(-1)^2 c_0}{2^2 \lfloor 2}$

$k = 3 \Rightarrow c_3 = -\frac{1}{6} c_2 = (-1)^3 \cdot \frac{1}{2^2 \lfloor 2} \times \frac{c_0}{6}$

$= \frac{(-1)^3 c_0}{2^3 \times 3!}$ etc

$c_k = \frac{(-1)^k c_0}{2^k \cdot k!}$

$\varphi_2(x) = x^{1/2} \sum_{k=0}^{\infty} \frac{(-1)^k}{2^k \lfloor k} x^k$

$\varphi_2(x) = x^{1/2} e^{-x/2}$ taking $c_0 = 1$.

Find a soln of the form $\varphi(x) = x^k \sum_{m=0}^{\infty} c_m x^{k+m}$ for the eqn $x^2 y'' + xy' + (x^2 - 1) y = 0$

Soln:

$L(y) = x^2 y'' + xy' + (x^2 - 1) y = 0 \quad \rightarrow (1)$

$\varphi(x) = \sum_{m=0}^{\infty} c_m x^{k+m} \quad c_0 \neq 0, \quad \rightarrow (2)$

$\varphi'(x) = \sum_{m=0}^{\infty} (k+m) c_m x^{k+m-1}$

$\varphi''(x) = \sum_{m=0}^{\infty} (k+m)(k+m-1) c_m x^{k+m-2}$

$$① \Rightarrow x^2 \sum_{m=0}^{\infty} (k+m)(k+m-1) C_m x^{k+m-2} + x \sum_{m=0}^{\infty} (k+m) C_m x^{k+m-1} + (x^2-1) \sum_{m=0}^{\infty} C_m x^{k+m} = 0$$

$$\sum_{m=0}^{\infty} (k+m)(k+m-1) C_m x^{k+m} + \sum_{m=0}^{\infty} (k+m) C_m x^{k+m} + \sum_{m=0}^{\infty} C_m x^{k+m+2} - \sum_{m=0}^{\infty} C_m x^{k+m+2} = 0$$

$$\Rightarrow \sum_{m=0}^{\infty} [(k+m)(k+m-1) + (k+m-1)] C_m x^{k+m} + \sum_{m=0}^{\infty} C_m x^{k+m+2} = 0$$

$$\sum_{m=0}^{\infty} [(k+m)^2-1] C_m x^{k+m} + \sum_{m=0}^{\infty} C_m x^{k+m+2} = 0 \rightarrow ③$$

equating to zero the coefficient of the smallest powers of x namely x^k we got the indicial eqn.

ie) $C_0 (k^2-1) = 0$

$C_0 (k-1)(k+1) = 0$

$k = 1, -1$ as $C_0 \neq 0$.

These are unequal and differ by an integer.

Here the diff in powers of x in ③ in ②.

Hence the we equate to zero the coefficient of x^{k+1} in

we obtain $[(k+1)^2-1] C_0 = 0$

$\Rightarrow k(k+2) C_1 = 0 \Rightarrow C_1 = 0$ for both $k = 1, -1$

equating to zero to coefficient of x^{k+m} in ③

we obtain $[(k+1)^2-1] C_0 = 0$

we obtain $(k+m+1)(k+m-1) C_m + C_{m-2} = 0$

$$\therefore C_m = \frac{-1}{(k+m-1)(k+m+1)} C_{m-2} \rightarrow ④$$

Putting $m = 3, 5, 7 \ldots$ in ④ and nothing that $C_1 = 0$

we get $C_1 = C_3 = C_5 = \ldots = 0$

Putting. $m = 2, 4, 6 ..$ in (5)

$$c_2 = \frac{-1}{(k+1)(k+3)} c_0 \rightarrow (5)$$

$$c_4 = \frac{-1}{(k+3)(k+5)} c_2$$

$$c_4 = \frac{1}{(k+1)(k+3)^2(k+5)} c_0 \quad \text{etc.}$$

$$\therefore y = \varphi(x) = c_0 x^k \left\{ 1 - \frac{x^2}{(k+1)(k+3)} + \frac{x^4}{(k+1)(k+3)^2(k+5)} + \cdots + \infty \right] \rightarrow (6)$$

If we take $k = -1$ in the above series the coefficient becomes infinite because of the factor $(k+1)$ in the denominator.

$\therefore$ Put $c_0 = d_0(k+1)$ in (6)

$$\varphi(x) = d_0 x^k \left\{ (k+1) - \frac{x^2}{k+3} + \frac{x^4}{(k+3)^2(k+5)} \cdots \infty \right\} \rightarrow (7)$$

Diff (7) Partially w.r.t k we get,

$$\frac{\partial \varphi}{\partial k} = d_0 x^k \log x \left\{ (k+1) - \frac{x^2}{k+3} + \frac{x^4}{(k+3)^2(k+5)} \cdots \infty \right\}$$

$$+ d_0 x^k \left[1 + \frac{x^2}{(k+3)^2} - \left\{ \frac{2}{(k+3)^2(k+5)} + \frac{1}{(k+3)^2(k+5)^2} \right\} x^4 \ldots \right\} \rightarrow (8)$$

Putting $k = -1$, $d_0 = a$ in (7) we get,

$$\varphi = a x^{-1} \left[-\frac{x^2}{2} + \frac{x^4}{2^2 \cdot 4} + \frac{x^6}{2^2 \cdot 4^2 \cdot 6} + \cdots \right] = a u \ (\text{say}) \rightarrow (9)$$

from (8), $k = -1$ where d_0 is replaced by b.

we get,

$$\frac{\partial y}{\partial k} = b(\log x) x^{-1} \left[-\frac{x^2}{2} + \frac{x^4}{2^2 \cdot 4} + \frac{x^6}{2^2 \cdot 4^2 \cdot 6} + \cdots \right]$$

$$+ b x^{-1} \left[1 + \frac{x^2}{2} - \frac{1}{2^2 \cdot 4} + \left(\frac{2}{2} + \frac{1}{4}\right) x^4 + \cdots \right]$$

(30)

ii) $\frac{\partial y}{\partial k} = b\log x . u + b x^{-1}\left[1 + \frac{x^2}{2^2} - \frac{1}{2^2 . 4} + \left(\frac{2}{2} + \frac{1}{4}\right)x^4 - 1\right]$

$= bu$ (say) $\rightarrow$ (10)

Putting $k=1$ in (9) we get,

$y = d_0 x\left[2 - \frac{x^2}{4} + \frac{x^4}{4^2 . 6} + \cdots\right]$

$y = -2^2 d_0 x^{-1}\left[-\frac{x^2}{2} + \frac{x^4}{2^2 . 4} - \frac{x^6}{2^2 . 4^2 . 6} + \cdots\right] \rightarrow$ (11)

from (9) and (11) we find all the two Soln are linearly dependent

Hence (9) and (10) are the required linearly independent Solns.

3. S.T -1 and 1 regular Singular

(a) Points for the legendre eqn $(1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0$

(b) Find all the indicial polynomial and its roots corresponding to the Point $x=1$, Aliter for (2) $x^2y'' + \frac{3}{2}xy' + xy = 0$

Soln:

Given $L(y) = x^2y'' + \frac{3}{2}xy' + xy = 0 \rightarrow$ (1)

$\varphi(x) = \sum_{k=0}^{\infty} c_k x^{r+k}$

$\varphi'(x) = \sum_{k=0}^{\infty} (r+k) c_k x^{r+k-1}$

$$(1) \Rightarrow \left.\begin{array}{l} x^2\sum_{k=0}^{\infty} c_k(r+k)(r+k-1)x^{r+k} + \frac{3}{2}\sum_{k=0}^{\infty} c_k(r+k)x^{r+k} \\ + \sum_{k=0}^{\infty} c_k x^{r+k+1} \end{array}\right\} = 0$$

$$\Rightarrow \left.\begin{array}{l} \sum_{k=0}^{\infty} c_k(r+k)(r+k-1)x^{r+k} + \frac{3}{2}\sum_{k=0}^{\infty} c_k(r+k)x^{r+k} \\ + \sum_{k=1}^{\infty} c_{k-1}x^{r+k} \end{array}\right\} = 0$$

$$\left.\begin{array}{l} c_0[r(r-1)x^r] + \sum_{k=1}^{\infty} c_k(r+k)(r+k-1)x^{r+k} + \frac{3}{2}c_0 r x^r \\ + \sum_{k=1}^{\infty} c_k(r+k)x^{r+k} \end{array}\right\} = 0$$

$$\left.\begin{aligned} C_0 r(r-1)x^r + \frac{3}{2} C_0 r x^r + \sum_{k=1}^{\infty}\Big[C_k(r+k)(r+k-1) \\ + \frac{3}{2} C_k(r+k) + C_{k-1}\Big] x^{r+k} \end{aligned}\right\} = 0$$

Equating coefficient of x^r

$$\Rightarrow C_0(r^2-r) + \frac{3}{2} r C_0 = 0$$

$$C_0\left[r^2 - r + \frac{3}{2} r\right] = 0$$

$\because C_0 \neq 0$,

$$r^2 - r + \frac{3}{2} r = 0$$

$$r^2 + \frac{r}{2} = 0$$

$$2r^2 + r = 0$$

$$\Rightarrow r(2r+1) = 0 \Rightarrow r = 0, r = -1/2$$

Since $q(r) = 0$

equating x^{r+k} terms we get,

$$C_k(r+k)(r+k-1) + \frac{3}{2} C_k(r+k) + C_{k-1} = 0$$

$$C_k(r+k)\left[(r+k-1) + \frac{3}{2}\right] = -C_{k-1}$$

$$C_k = \frac{-C_{k-1}}{(r+k)(r+k+\frac{1}{2})} = \frac{-C_{k-1}}{q(r+k)} \rightarrow ②$$

Case (i) when $r=0$

$$C_k = \frac{-C_{k-1}}{q(k)}, \quad k \geq 1$$

$k=1$, $C_1 = -\dfrac{C_0}{q(1)}$, $k=2$, $C_2 = -\dfrac{C_1}{q(2)} = \dfrac{C_0}{q(1)q(2)}$

$$q_1(x) = \sum_{k=0}^{\infty} C_k x^{r+k} = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \dots$$

$$= C_0 - \frac{C_0}{q(1)} x + \frac{C_0}{q(1)q(2)} x^2 + \dots$$

$$= C_0\left[1 - \frac{x}{q(1)} + \frac{x^2}{q(1)q(2)} \dots\right]$$

$$q_1(x) = \left[1 + \sum_{k=1}^{\infty} \frac{(-1)^k x^k}{q(1)q(2)\dots q(k)}\right]$$

taking $c_0 = 0$

Case (ii)

When $r = -\frac{1}{2}$

$$c_k = \frac{-c_{k-1}}{q(k-\frac{1}{2})}, \quad k \geq 1$$

$$k=1, \quad c_1 = -\frac{c_0}{q(\frac{1}{2})} \rightarrow k=2, \quad c_2 = -\frac{c_1}{q(\frac{3}{2})} = \frac{c_0}{q(\frac{1}{2})\,q(\frac{3}{2})}$$

$$\varphi_2(x) = x^r \cdot \sum_{k=0}^{\infty} c_k x^k$$

$$= x^{-\frac{1}{2}} [c_0 + c_1 x + c_2 x^2 + \dots]$$

$$= x^{-\frac{1}{2}} \left[c_0 - \frac{c_0}{q(\frac{1}{2})} x + \frac{c_0 x^2}{q(\frac{1}{2})q(\frac{3}{2})} + \dots \right]$$

$$= x^{-\frac{1}{2}} c_0 \left[1 - \frac{x}{q(\frac{1}{2})} + \frac{x^2}{q(\frac{1}{2})q(\frac{3}{2})} + \dots \right]$$

taking $c_0 = 1$,

$$\varphi_2(x) = x^{-\frac{1}{2}} \left[1 + \sum_{k=1}^{\infty} \frac{(-1)^k x^k}{q(1)\dots q(k-\frac{1}{2})} \right]$$

Hence the soln is

$$\varphi = A\varphi_1(x) + B\varphi_2(x)$$

$$\varphi = 1 + \sum \frac{(-1)^k x^k}{q(1)q(2)\dots q(k)} + Bx^{-\frac{1}{2}} \sum_{n=1}^{\infty} \frac{(-1)^k x^k}{q(1)\dots q(k-\frac{1}{2})}$$

Second order eqn with regular singular points

The General Case:

Suppose we have a soln φ of the form.

$$\varphi(x) = x^r \sum_{k=0}^{\infty} c_k x^k \quad (c_0 \neq 0) \rightarrow (1)$$

for the eqn $x^2 y'' + a(x) x y' + b(x) y = 0 \rightarrow (2)$

where
$$\left.\begin{aligned} a(x) &= \sum_{k=0}^{\infty} \alpha_k x^k \\ b(x) &= \sum_{k=0}^{\infty} \beta_k x^k \end{aligned}\right\} \rightarrow (3)$$

for $|x|<0$.

Then $\phi'(x) = \sum_{K=0}^{\infty} (K+r) C_K x^{K+r-1}$

$$\phi'(x) = x^{r-1} \sum_{K=0}^{\infty} (K+r) C_K x^K$$

$$\phi''(x) = \sum_{K=0}^{\infty} (K+r)(K+r-1) C_K x^{K+r-2}$$

$$= x^{r-2} \sum_{K=0}^{\infty} (K+r)(K+r-1) C_K x^K$$

and hence.

$$b(x)\,\phi(x) = x^r \left[\sum_{K=0}^{\infty} C_K x^K\right]\left[\sum_{K=0}^{\infty} \beta_K x^K\right]$$

$$= x^r \left[\sum_{K=0}^{\infty} \bar{\beta}_K x^K\right] \qquad \therefore \bar{\beta}_K = \sum_{j=0}^{\infty} \beta_K x^K$$

$$x.a(x).\phi'(x) = x^r \left[\sum_{K=0}^{\infty} (K+r) C_K x^K\right]\left[\sum_{K=0}^{\infty} \alpha_K x^K\right]$$

$$= x^r \sum_{K=0}^{\infty} \bar{\alpha}_K x^K \qquad \therefore \bar{\alpha}_K = \sum_{j=0}^{\infty} (j+r) C_j \alpha_{K-j}$$

$$x^2 \phi''(x) = x^r \sum_{K=0}^{\infty} (K+r)(K+r-1) C_K x^K$$

Thus ② becomes,

$$L[\phi(x)] = x^r \sum_{K=0}^{\infty} \left[(K+r)(K+r-1) C_K + \bar{\alpha}_K + \bar{\beta}_K\right] x^K$$

and we must have,

$$L[\phi(x)]_K = [(K+r)(K+r-1) C_K + \bar{\alpha}_K + \bar{\beta}_K] = 0. \quad K = 0,1,2\ldots$$

using the defn of $\bar{\alpha}_K$ and $\bar{\beta}_K$ $[L(\phi(x))]_K$ as

$$L[\phi(x)]_K = (K+r)(K+r-1) C_K + \sum_{j=0}^{\infty} (j+r) C_j \alpha_{K-j} + \sum_{j=0}^{\infty} C_j \beta_{K-j}$$

$$= [(K+r)(K+r-1) + (K+r)\alpha_0 + \beta_0] C_K + \sum_{j=0}^{K-1} [(j+r)\alpha_{K-j} + \beta_{K-j}] C_j$$

for $K=0$ we have $r(r-1) + r\alpha_0 + \beta_0 = 0$ $\quad \therefore C_0 \neq 0$.

The Second degree Polynomial q given by.

$q(r) = r(r-1) + r(\alpha_0) + \beta_0$ is called the indical Polynomial for ②

The only admissible values of 'r' are the roots of q. (3)

we see that.

$$L[\varphi(x)]_k = q(r+1)\,c_k + d_k = 0, \quad k=1,2\ldots \rightarrow (5)$$

where $d_k = \sum_{j=0}^{k-1}[(j+r)\,\alpha_{k-j} + \beta_{k-j}]\,c_j, \quad k=1,2\ldots \rightarrow (6)$

We observe that d_k is a linear combination of $c_0, c_1, \ldots c_k$ with the coefficients involving the known functions a, b and

Let us solve (5) and (6) successively, interms of c_0 and let $c_k(r)$ denote the solns and the corresponding d_k by $D_k(r)$

Thus $D_1(r) = (r\alpha_1 + \beta)\,c_0$

$$c_1(r) = -\frac{D_1(r)}{q(r+1)}$$

an ingeneral

$$D_k(r) = \sum_{j=0}^{k-1}[(j+r)\,\alpha_{k-j} + \beta_{k-j}]\,c_j(r)$$

$$c_k(r) = -\frac{D_k(r)}{q(r+k)}, \quad k=1,2\ldots \rightarrow (7)$$

The c_k, thus determined are rational functions of r and the only points where they does not exist are the points for which $q(r+k)=0$ for some $k=1,2\ldots$

only two such possible points exists

Let $\bar{\varphi}(x,r) = c_0 x^r + x^r \sum_{k=1}^{\infty} c_k(r)\,x^k \rightarrow (8)$

If the series (8) converges for $0<x<r_0$ then clearly

$$L(\bar{\varphi})(x,r) = c_0\,q(r)\,x^r \rightarrow (9)$$

e now have the following

If the ϕ given by ① is a soln ② then r must be a root of indicial polynomial q and the c_k $(k \geq 1)$ are determined uniquely in terms of r and c_0 by be the $c_k(r)$ of ⑦ provided $q(r+k) \neq 0$, $k = 1, 2 \ldots$

Conversely,

If r is a root of q, and if the $c_k(r)$ can be determined.

(i) $q(r+k) \neq 0$, $k = 1, 2 \ldots$

Then the fun ϕ given by

$\phi(x) = \bar{\phi}(x, r)$ is a soln of ② for any choice of c_0 provided the series ⑧ converges.

Let, r_1, r_2 be the two roots of $[q(r)]$ and let $Re\, r_1 \geq Re\, r_2$

Then $q(r_1+k) \neq 0$ for any $k = 1, 2 \ldots$

Thus we have $\phi_1(x) = x^{r_1} \sum_{k=0}^{\infty} c_1(r_1) x^k$

$c_0(r_1) = 1$ → ⑩ as a soln of ②

Provided the series is convergent.

If r_2 is a root of q distinct from r_1 and $q(r_2+k) \neq 0$ for $k = 1, 2 \ldots$ Then clearly,

$$\phi_2(x) = x^{r_2} \sum_{k=0}^{\infty} c_k(r_2) \cdot x^r, \quad c_0(r_2) = 1$$

→ ②

is another soln of ② provided the series is convergent.

Note:

The condition $q(r_2+k) \neq 0$ for $k = 1, 2 \ldots$ is the same as $r_1 \neq r_2 + k$ for $k = 1, 2 \ldots$ (or) $r_1 - r_2$ is not a positive integer.

Since $\alpha_0 = a(0)$, $\beta_0 = b(0)$. The indicial polynomial $q(r)$ can be written as

$$q(r) = r(r-1) + a(0) + b(0).$$

Theorem:

Consider the eqn $x^2y''+a(x)xy'+b(x)y=0$. Where a, b have convergent power series expansion for $|x|<r_0$, $r_0>0$.

Let r_1, r_2 ($Re\, r_1 \geq Re\, r_2$) be the roots of the indicial Polynomial $q(r)=r(r-1)+a(0)+b(0)$

for $0<|x|<r_0$ There is a soln φ_1 of the form,

$$\varphi_1(x)=|x|^{r_1}\sum_{k=0}^{\infty}c_k x^k \qquad (\because c_0 = 1)$$

Where the Series converges for $|x|<r_0$.

If r_1-r_2 is not zero on a positive integer there a second solns φ_2 for $0<|x|<r_0$ of the form,

$$\varphi_2(x)=|x|^{r_2}\sum_{k=0}^{\infty}c_k x^k$$

Where the series converges for $|x|<r_0$ the coeff $c_k, \bar{c}_k$ can be obtained by substitution of the soln in to the differential eqn.

Remark:

(i) The coefficient $c_k, \bar{c}_k$ appearing in the soln's φ_1, φ_2 of the above theorem are given by

$$c_k = c_k(r_1), \quad \bar{c}_k = c_k(r_2), \quad k=0,1,2\ldots$$

where $c_k(r) \Rightarrow k=1,2\ldots$ are soln of the equations ① and ②

with $c_0(r)=1$

(ii) If r_1-r_2 is either or a positive integer we say that we have an exceptional case.